An Analysis of Vibrations of Quartz Crystal Plates with Nonlinear Mindlin Plate Equations

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Abstract*—The nonlinear effects of material constants and initial stresses and strains in quartz crystal resonators is well known f on the frequency-temperature curves, drive-level dependency, acceleration sensitivity, and stress compensation. Consequently, accurate predictions on resonator behavior and their electrical circuit parameters require the use of nonlinear vibration equations and their solutions. The effectiveness of nonlinear analyses has been shown by a few researchers with the finite element and perturbation methods. The Mindlin plate theory, which has been used extensively for understanding plate modes and their coupling effects in plate vibrations analysis, is not enough in the study of the nonlinear behavior of quartz resonators. We have followed the Mindlin plate theory to derive the nonlinear equations with the inclusion of large displacements and higher order elastic constants. The coupling of vibration modes due to nonlinearity is clearly observed and it is quite different from linear cases that we are familiar with. We start from the equations of vibration for the thickness-shear mode to validate the solution techniques, which could be the perturbation method and the latest Homotopy Analytical Method (HAM). Then the methods are applied to the coupled equations of thickness-shear and flexural vibrations which are the two dominant modes of quartz crystal resonators of the thickness-shear type. These solutions, in the absence of the strong electrical field, can be used to study the frequency, deformation, and mode conversion in nonlinear vibrations. We hope the frequency spectra and spatial variations of the thickness-shear and flexural displacements from the accurate solutions of nonlinear equations will provide insights on the changes in each mode when compared with their linear vibrations. The further extension of nonlinear plate equations with the inclusion of piezoelectric effects will also provide useful examination of nonlinear behavior of quartz crystal resonators.

I. Introduction

The linear Mindlin plate equations have been widely used in the analysis of quartz crystal resonators for vibration frequency analysis and mode shape prediction as the only available two-dimensional theory [1-4]. It has also been implemented in the finite element analysis to accommodate considerations of complicated structures and biasing fields [5-7]. The results, both analytical and numerical, have been playing important roles in the design process of quartz crystal resonators and further efforts are still devoted to make the analysis more accurate, tools more convenient, and cost low to meet development needs.

Meanwhile, as the analysis of quartz crystal resonators advances toward estimating resonator properties and precise design is in strong demand, the nonlinear behavior which has been puzzling the quartz crystal resonator industry is also getting the deserved attention. For instance, persistent studies and accumulated knowledge have been revealing that drive-level dependence (DLD) and activity-dip are related to the nonlinear properties of quartz crystal and they should be studied for the improvement of resonator design and applications. There are research work concerning the nonlinear properties of quartz crystal resonator and analytical methods [8-11] and numerical methods such as the finite element analysis has been utilized. These studies addressed the concern of nonlinear effects in quartz crystal resonators and showed important findings on possible improvement remedies in design and manufacturing processes. As a result, we feel that a systematic study of the nonlinear effect of quartz crystal resonators are needed for better understanding of the combined effect and provide design guidelines with these studies.

As part of the effort to understand the nonlinear effect and provide analytical methods, we start from the derivation of nonlinear the Mindlin plate equations for the analysis of high frequency vibrations of piezoelectric plates. The reason we take this approach is that the Mindlin plate equations have been effective and earlier methods and procedures can be directly applicable if nonlinear equation are also established in parallel. What we shall need to consider include the nonlinear material properties, which are the higher-order material constants, and, of course, the nonlinear strain components [12]. As a result, the equations are changed with increased couplings of vibration modes through the nonlinear constitutive relations. As we know, this will make the equations much more complicated and the solution technique will be different. As nonlinear equations, newer solution methods have to be sought for these new problems. On the

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other hand, these solutions will be used for the study of vibration frequency and mode shapes, which are essential in the calculation of resonator properties for circuit applications with emphasis on the nonlinear behavior we are more concerned for precision type products. Actually, the motivation of our study on nonlinear vibrations of quartz crystal resonators is to provide analytical tools to investigate the nonlinear behavior of resonators which are important in many vital applications. Among them, the traditional DLD phenomenon is one of them to be studied with nonlinear material properties [10]. Recent studies on the nonlinear effects of quartz crystal resonators and the introduction of the nonlinear theory for the analysis of piezoelectric solids prompted us to study the problem based on the Mindlin plate theory, which has been the cornerstone for analysis and design since its establishment. This effort can be treated as an extension of successful applications of the Mindlin plate theory for the quartz crystal resonator technology.

II. Nonlinear Mindlin Plate Equations

The Mindlin plate theory starts from the power series expansion of displacements in the thickness coordinate of a typical plate shown in Fig. 1 in the form of

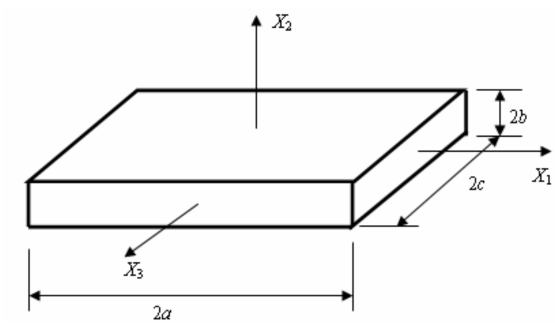

Fig. 1 A typical plate with coordinate system.

$$u_j(x_1,x_2,x_3,t)=\sum_{n=0}^{\infty}u_j^{(n)}(x_1,x_3,t)x_2^n,\ j=1,2,3, \tag{1}$$

where $u_j^{(n)}, x_j$ and t are the nth-order displacements, coordinates, and time, respectively.

The strain components with the consideration of nonlinear terms will have the form of [12]

$$\begin{aligned} S_{kl} &= \frac{1}{2}\sum_{n=0}^{\infty}\left[u_{k,l}^{(n)}+\delta_{2l}(n+1)u_k^{(n+1)}+u_{l,k}^{(n)}+\delta_{2k}(n+1)u_l^{(n+1)}\right]x_2^n \\ &+\frac{1}{2}\sum_{s=0}^{\infty}\left[u_{m,k}^{(s)}+\delta_{2k}(s+1)u_m^{(s+1)}\right]x_2^s\cdot\sum_{t=0}^{\infty}\left[u_{m,l}^{(t)}+\delta_{2l}(t+1)u_m^{(t+1)}\right]x_2^t \\ &=\sum_{n=0}^{\infty}S_{kl}^{(n)}x_2^n, k,l=1,2,3, \end{aligned} \tag{2}$$

where

$$\begin{aligned} S_{kl}^{(n)} &= \frac{1}{2}\left[u_{k,l}^{(n)}+u_{l,k}^{(n)}+(n+1)\left(\delta_{2l}u_k^{(n+1)}+\delta_{2k}u_l^{(n+1)}\right)\right] \\ &+\frac{1}{2}\left\{\sum_{s}\sum_{t}^{s+t=n}\left[u_{m,l}^{(s)}+\delta_{2l}(s+1)u_m^{(s+1)}\right]\cdot\left[u_{m,k}^{(t)}+\delta_{2k}(t+1)u_m^{(t+1)}\right]\right\}. \end{aligned} \tag{3}$$

is the nth-order strain component.

Now the constitutive relations will take the form of

$$T_{ij}=\sum_{n=0}^{\infty}\left(c_{ijkl}S_{kl}^{(n)}+\frac{1}{2}c_{ijklmn}S_{klmn}^{(n)}\right)x_2^n, i,j,k,l,m,n=1,2,3, \tag{4}$$

where the fourth-order strain tensor is defined as

$$S_{klmn}^{(n)}=\sum_{s}\sum_{t}^{s+t=n}S_{kl}^{(s)}S_{mn}^{(t)}. \tag{5}$$

For demonstration purpose, we can expand the zeroth- and first-order components of the fourth-order strain tensor to

$$S_{klmn}^{(0)}=S_{kl}^{(0)}\cdot S_{mn}^{(0)}, S_{klmn}^{(1)}=S_{kl}^{(1)}\cdot S_{mn}^{(0)}+S_{kl}^{(0)}\cdot S_{mn}^{(1)}. \tag{6}$$

With the three-dimensional variational equation of motion of elasticity in an elastic solid with volume V,

$$\int_V\left(T_{ij,i}-\rho\ddot{u}_j\right)\delta u_j dV=0, \tag{7}$$

where ρ is the density of the material, and the displacements in (1) and stress in (4), we now have

$$\int_A\int_{-b}^{b}\left(T_{ij,i}-\rho\sum_{n=0}^{\infty}\ddot{u}_j^{(n)}x_2^n\right)\sum_{n=0}^{\infty}\delta u_j^{(n)}x_2^n dx_2 dA=0, \tag{8}$$

where A and b are the faces and half thickness of the plate shown in Fig. 1, respectively.

By carrying out the integration over the thickness coordinate, we now have

$$\int_A\sum_{n=0}^{\infty}\left(T_{ij,i}^{(n)}-nT_{2j}^{(n-1)}+F_j^{(n)}-\rho\sum_{n=0}^{\infty}B_{mn}\ddot{u}_j^{(m)}\right)\delta u_j^{(n)}dA=0, \tag{9}$$

where

$$\begin{aligned} T_{ij}^{(n)} &= \sum_{m=0}^{\infty}B_{mn}\left(c_{ijkl}S_{kl}^{(m)}+\frac{1}{2}c_{ijklmn}S_{klmn}^{(m)}\right), \\ F_j^{(n)} &= b^nT_{2j}(b)-(-b)^nT_{2j}(-b), \\ B_{mn} &= \begin{cases} \dfrac{2b^{m+n+1}}{m+n+1}, & m+n \text{ even}, \\ 0, & m+n \text{ odd}. \end{cases} \end{aligned} \tag{10}$$

For independent variation of the nth-order displacement $\delta u_j^{(n)}$, we must have the nth-order displacement equations of motion as

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho \sum_{n=0}^{\infty} B_{mn} \ddot{u}_j^{(m)}. \tag{11}$$

It is clear that (11) is precisely the *n*th-order equations of motion of the linear Mindlin plate theory. The nonlinear terms, as we can see, is embedded in the strain terms in (10).

By expanding the above equations with displacements, we can obtain the successive order of strain and stress components from (3) and (10). The difference between the linear and nonlinear components is in the nonlinear strain terms and material constants. By substituting these stress components into the two-dimensional equations of motion in (11), we shall have a series of higher-order equations of motion in two-dimensional variables.

Finally, the boundary conditions for the nonlinear plate equations will be the same as the linear Mindlin plate equations. We only need to specify the two-dimensional tractions or displacement components but not both on the cylindrical surfaces.

For demonstration purpose, the first-order equations of the nonlinear Mindlin plate equation are obtained with a simple truncation procedure by letting

$$u_j^{(n)} = S_{ij}^{(n)} = T_{ij}^{(n)} = S_{ijkl}^{(n)} = 0, \text{for } n > 1. \tag{12}$$

Sophisticated truncation procedure suggested by Mindlin can also be performed [1, 3-4], but there will be approximations due to the presence of coupled displacements and their derivatives. We leave the discussion of the truncation procedure of nonlinear equations for future treatment. Consequently, we have the first-order equations of motion as

$$\begin{aligned}
&T_{11,1}^{(0)} + T_{13,3}^{(0)} + F_1^{(0)} = 2b\rho\ddot{u}_1^{(0)},\\
&T_{12,1}^{(0)} + T_{23,3}^{(0)} + F_2^{(0)} = 2b\rho\ddot{u}_2^{(0)},\\
&T_{13,1}^{(0)} + T_{33,3}^{(0)} + F_3^{(0)} = 2b\rho\ddot{u}_3^{(0)},\\
&T_{11,1}^{(1)} + T_{13,3}^{(1)} - T_{12}^{(0)} + F_1^{(1)} = \frac{2b^3}{3}\rho\ddot{u}_1^{(1)},\\
&T_{12,1}^{(1)} + T_{23,3}^{(1)} - T_{22}^{(0)} + F_2^{(1)} = \frac{2b^3}{3}\rho\ddot{u}_2^{(1)},\\
&T_{13,1}^{(1)} + T_{33,3}^{(1)} - T_{23}^{(0)} + F_3^{(1)} = \frac{2b^3}{3}\rho\ddot{u}_3^{(1)}.
\end{aligned} \tag{13}$$

The zeroth- and first-order stress components are

$$\begin{aligned}
T_{ij}^{(0)} &= 2bc_{ijkl}\kappa_{ij}^{(0)}\kappa_{kl}^{(0)}S_{kl}^{(0)} + bc_{ijklmn}S_{klmn}^{(0)},\\
T_{ij}^{(1)} &= \frac{2b^3}{3}c_{ijkl}\kappa_{ij}^{(0)}\kappa_{kl}^{(0)}S_{kl}^{(1)} + \frac{b^3}{3}c_{ijklmn}S_{klmn}^{(1)}.
\end{aligned} \tag{14}$$

where $\kappa_{ij}^{(0)}$ and $\kappa_{kl}^{(0)}$ are correction factors [1, 3-4]. It should be emphasized that correction factors are adopted from Minldin without considering the nonlinear effect. The correction based on the nonlinear equations is worth to investigate, but we also leave this for future consideration.

It is well-known from linear equations that the thickness-shear and flexural modes are strongly coupled. Analysis of vibrations of these two coupled modes will provide important design guidelines for quartz crystal resonators. Naturally, we should also start our study of nonlinear vibrations of crystal plates from these two important modes. For this purpose, we have equations for thickness-shear and flexural modes as [12]

$$\begin{aligned}
&T_{12,1}^{(0)} = 2b\rho\ddot{u}_2^{(0)},\\
&T_{11,1}^{(1)} - T_{12}^{(0)} = \frac{2b^3}{3}\rho\ddot{u}_1^{(1)};\\
&T_{12}^{(0)} = 2bc_{1212}\kappa_{12}^{(0)}\kappa_{12}^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right) + \frac{b}{2}c_{121112}\left(u_{2,1}^{(0)}u_{2,1}^{(0)}\right)\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\\
&\qquad + \frac{b}{2}c_{122212}\left(u_1^{(1)}u_1^{(1)}\right)\left(u_{2,1}^{(0)} + u_1^{(1)}\right),\\
&T_{11}^{(1)} = \frac{2b^3}{3}c_{1111}\kappa_{11}^{(1)}\kappa_{11}^{(1)}u_{1,1}^{(1)} + \frac{2b^3}{3}c_{111111}\left(u_{2,1}^{(0)}u_{2,1}^{(0)}\right)u_{1,1}^{(1)}\\
&\qquad + \frac{b^3}{6}c_{111122}u_{1,1}^{(1)}\left(u_1^{(1)}u_1^{(1)}\right) + \frac{2b^3}{3}c_{111212}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\left(u_1^{(1)}u_{1,1}^{(1)}\right).
\end{aligned} \tag{15}$$

By substituting stress components into the stress equations of motion, we have the displacement equations of motion as

$$\begin{aligned}
&2bc_{1212}\left(\kappa_{12}^{(0)}\right)^2\left(u_{2,11}^{(0)} + u_{1,1}^{(1)}\right) + \frac{b}{2}c_{121112}\left[2\left(u_{2,11}^{(0)}u_{2,1}^{(0)}\right)\left(u_{2,1}^{(0)} + u_1^{(1)}\right) + \left(u_{2,1}^{(0)}u_{2,1}^{(0)}\right)\left(u_{2,11}^{(0)} + u_{1,1}^{(1)}\right)\right]\\
&\quad + \frac{b}{2}c_{122212}\left[2\left(u_{1,1}^{(1)}u_1^{(1)}\right)\left(u_{2,1}^{(0)} + u_1^{(1)}\right) + \left(u_1^{(1)}\right)^2\left(u_{2,11}^{(0)} + u_{1,1}^{(1)}\right)\right] = 2b\rho\ddot{u}_2^{(0)},\\
&\frac{2b^3}{3}c_{1111}\left(\kappa_{11}^{(1)}\right)^2u_{1,11}^{(1)} + \frac{2b^3}{3}c_{111111}\left[2\left(u_{2,11}^{(0)}u_{2,1}^{(0)}\right)u_{1,1}^{(1)} + \left(u_{2,1}^{(0)}\right)^2u_{1,11}^{(1)}\right] + \frac{b^3}{6}c_{111122}\left[2u_{1,1}^{(1)}\left(u_{1,1}^{(1)}u_1^{(1)}\right) + u_{1,11}^{(1)}\left(u_1^{(1)}\right)^2\right]\\
&\quad + \frac{2b^3}{3}c_{111212}\left[\left(u_{2,11}^{(0)} + u_{1,1}^{(1)}\right)\left(u_1^{(1)}u_{1,1}^{(1)}\right) + \left(u_{2,1}^{(0)} + u_1^{(1)}\right)\left(u_{1,1}^{(1)}\right)^2 + \left(u_{2,1}^{(0)} + u_1^{(1)}\right)\left(u_1^{(1)}u_{1,11}^{(1)}\right)\right]\\
&\quad - 2bc_{1212}\left(\kappa_{12}^{(0)}\right)^2\left(u_{2,1}^{(0)} + u_1^{(1)}\right) - \frac{b}{2}c_{121112}\left(u_{2,1}^{(0)}\right)^2\left(u_{2,1}^{(0)} + u_1^{(1)}\right) - \frac{b}{2}c_{122212}\left(u_1^{(1)}\right)^2\left(u_{2,1}^{(0)} + u_1^{(1)}\right) = \frac{2b^3}{3}\rho\ddot{u}_1^{(1)}.
\end{aligned} \tag{16}$$

We shall now need to find solutions for (16) for the applications of nonlinear theory.

III. Solutions of the Nonlinear Mindlin Plate Equations with the Finite Difference Method

Now we have the nonlinear plate equations of vibrations of two strongly coupled modes to solve. The seemingly complex equations are originated from the linear vibrations of plates, but they are typical nonlinear equation frequently encounter in many engineering fields. The solutions of such nonlinear equations, without exception, have been the focus of extensive studies with many techniques and methods available. A direct examination of (16) will not provide clear clues on possible solutions and applicable techniques. Extensive efforts have to be made for suitable method we can utilize. We have been paying attention to the newly emerged homotopy analysis method (HAM), which has been capable to solve a large class of nonlinear equations in many fields [13-14]. However, there are difficulties in applying the HAM at this stage, and we have to find other methods which can be effectively used for the solutions of the simple equations. The search with limited choices ends up with the finite difference method (FDM), a traditional method for the nonlinear differential equations, as our current choice. With the given simple geometry in Fig. 1 and homogeneous material, the implementation of the FDM is straightforward.

The AT-cut quartz crystal material constants, including the nonlinear ones, are available from Yang [15].

First, we solve the linearly coupled thickness-shear and flexural vibration equations from (16) to validate the FDM scheme and procedure [16-17]. The rectangular plate in Fig. 1 is divided into grids in the length direction x_1, and the time is also divided into small intervals. In the solution phase, we have the total numbers of grids of the length and time as

$$N_x = 200, N_t = 1000. \tag{17}$$

Then the FDM solutions for the linear equations are compared with the analytical solutions with satisfactory agreement. This proves that the FDM can be effectively used for the nonlinear problems involving the equations above.

Now with the same grids for the domain and time, we calculated displacements of the plate with the same set of parameters. The initial conditions are specified with the assumption that the displacements take the form of

$$\begin{aligned} &u_2^{(0)}(x_1,0) = A_0 \sin\frac{\pi}{2b}x_1, \\ &u_1^{(1)}(x_1,0) = B_0 \cos\frac{\pi}{2b}x_1, \\ &\frac{\partial u_1^{(1)}}{\partial t}(x_1,0) = 0, \frac{\partial u_2^{(0)}}{\partial t}(x_1,0) = 0, \end{aligned} \tag{18}$$

and the boundary conditions are

$$\begin{aligned} &T_{12}^{(0)}(x_1 = 0) = T_{11}^{(1)}(x_1 = 0) = 0, \\ &T_{12}^{(0)}(x_1 = 2a) = T_{11}^{(1)}(x_1 = 2a) = 0. \end{aligned} \tag{19}$$

Finally, the solutions are obtained with the initial amplitude ratio $B_0/A_0 = 4$.

Through the FDM calculation, we have displacements of given points at different time. These displacements will determine the deformation of plate for given time of one period. The displacements are in numerical values, but they can be used in the future for possible calculation of resonator properties. The flexural displacement at $B_0/A_0 = 4, x_1 = a/2$ is plotted in Fig. 2 from FDM solutions with $B_0/A_0 = 4$. The thickness-shear displacements at $x_1 = a/2$ are plotted in Fig. 3. The nonlinear effect of can be clearly seen from the solution of the thickness-shear displacements.

IV. Conclusions

With the finite difference method, we have successfully solved the nonlinear equations of the coupled thickness-shear and flexural vibrations of a rectangular quartz crystal plate. The difference between linear and nonlinear equations can be clearly seen from displacement solutions. These solutions, as intended, can be used for the study of nonlinear effects on vibrations of quartz crystal resonators with the calculation of device properties and performance indicators. Since the continuing and accelerated shrinkage of resonator size has amplified the effects of structural complications and biasing fields, the nonlinear equations and solutions will be important in precise analysis and design of quartz crystal resonators. The FDM has been proven as an effective method to obtain the solutions of free vibrations, and the improvement of solution technique will provide needed results for the analysis of nonlinear effects in a quartz crystal resonator.

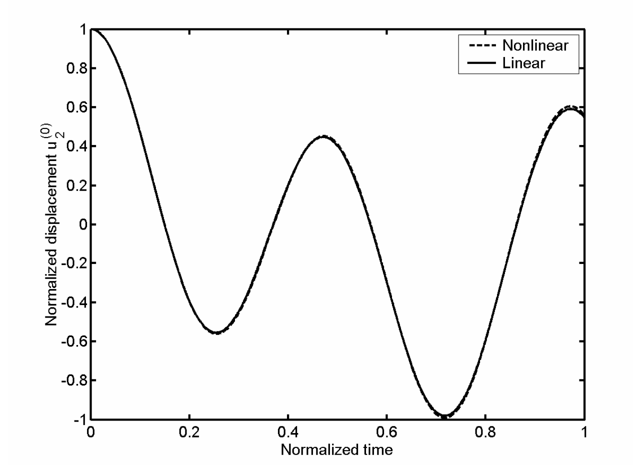

Fig. 2 Flexural displacements of a rectangular plate with linear and nonlinear equations at $x_1 = -a/2$ from the FDM solutions with $B_0/A_0 = 4$.

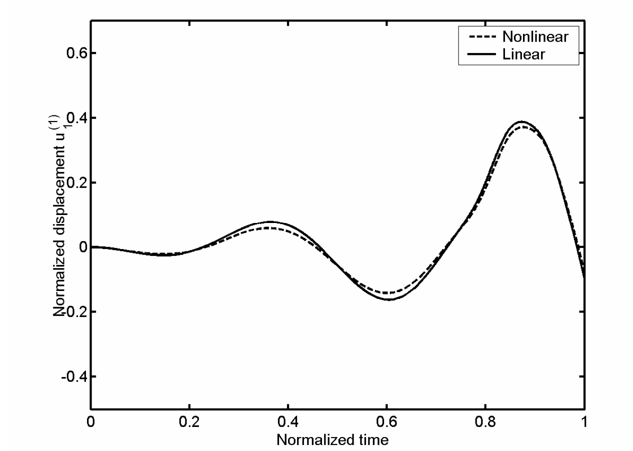

Fig. 3 Thickness-shear displacements of a rectangular plate with linear and nonlinear equations at $x_1 = -a/2$ from the FDM solutions with $B_0/A_0 = 4$.

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